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On p-Groups of Maximal Class, II

ANTONIO VERA-LÓPEZ* AND GUSTAVO A. FERNÁNDEZ-ALCOBER

Departamento de Matemáticas, Facultad de Ciencias, Universidad del País Vasco, Apartado 644, Bilbao, Spain

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INTRODUCTION

In this paper, we continue the study of the p-groups of maximal class from the point of view of the number of conjugacy classes, which was begun in [16] by A. Vera-López and B. Larrea. A p-group G of order p^m is said to be of maximal class if its nilpotency class is $m-1$. Unless otherwise stated, it is supposed throughout this paper that $m \geqslant 4$. The notation used here is practically the same as in [16]. Furthermore, we introduce the vector

$$\nabla_G = (a_0, a_1, ..., a_{m-2}),$$

where a_i is the number of conjugacy classes of order p^i for $0 \leqslant i \leqslant m-2$. We set $z_i = r_{Y_i}(s_i Y_{i+1})$ for $1 \leqslant i \leqslant m-1$ and $z_0 = r_G(sY_1)$. Thus, $\sigma_G = (z_0, z_1, ..., z_{m-1})$. If $\bar{G}$ is a quotient group of G, we write $\bar{z}_i$ for the $(i+1)$st component of the vector $\sigma_{\bar{G}}$. For the sake of briefness, we will represent the vector $(p^n, p^{n-1}, ..., p, 1)$ by the symbol τ_n. We will say that the following characteristic series of G,

$$G = Y_0 > Y_1 > \cdots > Y_{m-1} > Y_m = 1,$$

is the *extended lower central series* of G. Also, we call *$\mathscr{M}$-chain of length s* of a p-group of maximal class G, a series

$$G = H_0 > H_1 > \cdots > H_s,$$

in which $s \leqslant m-3$ and each H_i is a p-group of maximal class and a maximal subgroup of the corresponding H_{i-1}.

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If G is a p-group of order $p^m = p^{2n+e}$, P. Hall proves in [4] the existence of an integer $k = k(G) \geqslant 0$ such that

$$r(G) = n(p^2 - 1) + p^e + k(p^2 - 1)(p - 1).$$

We then write $r(G) = f_k(|G|)$. In the first section, we are mostly concerned with the study of $k(G)$ when G is a p-group of maximal class, especially with the relations between this and other invariants of G, such as m, $c(G)$, and σ_G. We wish to underline the following results:

(A) We prove the equality

$$\frac{z_0}{p} + \frac{z_1}{p^2} + \cdots + \frac{z_{n-1}}{p^n} = n + k(p-1),$$

which relates $k(G)$ with the local invariants z_i for $0 \leqslant i \leqslant n-1$.

(B) We determine a lower bound $B_{m,p}(c(G))$ for $k(G)$ in terms of p, m, and $c(G)$, which is best possible. Furthermore, we state the conditions on the commutator subgroups of G and on the numbers z_i $(0 < 2i \leqslant m - c(G) - 1)$ under which $k(G)$ coincides with the above-mentioned lower bound.

(C) If Y_1 is non-abelian and $k(G) > 0$, we prove that $c(G) \leqslant \log_p k + 2$, which improves the inequality $c(G) \leqslant \log_p k + 3$ given in [16].

(D) We show that, if $m \geqslant 2k + 5$, then $c(G) \leqslant 1$.

(E) In [14], J. Poland states that, if $k(G) = 0$, then $m \leqslant p + 2$. In this paper, we consider the general case and prove that, if $k(G) < F_\alpha(p) = p^\alpha + \cdots + p + 1$, then

$$m \leqslant 2p + \alpha(p-1) - c(G)(p-2),$$

whence we also obtain $c(G) \leqslant 2 + \alpha(p-1)/(p-2)$. As an example, if $1 \leqslant k(G) \leqslant p$, we get that $m \leqslant 2p + 1$.

(F) We give recursive formulas for the two sequences $\{k(H_i)\}_{i=0}^{m-3}$ and $\{k(Y_i)\}_{i=0}^{m}$ in terms of the invariants z_i.

In Sections 2 and 3, where we consider the p-groups of maximal class satisfying $k(G) = 0$ and 1, respectively, we shall use extensively the results of the first section. For $k(G) = 0$, we complete the information given in [14, 16] by determining all the commutator subgroups $[Y_i, Y_j]$ and the terms of the sequences $\{k(H_i)\}$ and $\{k(Y_i)\}$. For the case $k(G) = 1$, which is studied for the first time in this paper, we find σ_G, the commutator subgroups and the two sequences $\{k(H_i)\}$ and $\{k(Y_i)\}$ in terms of some suitable group invariants which arise in a natural way when dealing with these groups. Furthermore, we state Theorems (3.10) and (3.11), from which one can derive the vector ∇_G.

1. General Results

(1.1) Lemma. *Let G be a p-group of maximal class of order p^m with $m = 2n + e$ and $e \in \{0, 1\}$. Set $r(G) = f_k(|G|)$. Then, $\sigma_G = (z_0, z_1, ..., z_{n-1}, \tau_{n-1+e})$ and*

$$\frac{z_0}{p} + \frac{z_1}{p^2} + \cdots + \frac{z_{n-1}}{p^n} = n + k(p-1). \tag{1}$$

Proof. We have $\sigma_G = (z_0, z_1, ..., z_{m-1})$. If $i \in \{n, ..., m-1\}$, then $[Y_i, Y_i] = [Y_i, Y_{i+1}] \leqslant Y_{2i+1} \leqslant Y_{2n+1} = 1$. Hence, Y_i is abelian and $z_i = r_{Y_i}(s_i Y_{i+1}) = |s_i Y_{i+1}| = p^{m-i-1}$. We deduce that $(z_n, z_{n+1}, ..., z_{m-1}) = (p^{n-1+e}, p^{n-2+e}, ..., 1) = \tau_{n-1+e}$, whence $\sigma_G = (z_0, z_1, ..., z_{n-1}, \tau_{n-1+e})$.

On the other hand, arguing as in [2, Note E], we get

$$\begin{aligned}\frac{z_0}{p} + \frac{z_1}{p^2} + \cdots + \frac{z_{n-1}}{p^n} + \frac{z_n}{p^{n+1}} + \cdots + \frac{z_{m-1}}{p^m} + \frac{1}{p^m(p^2-1)} \\ = \frac{r(G)}{p^2-1} = \frac{f_k(|G|)}{p^2-1} = n + \frac{p^e}{p^2-1} + k(p-1).\end{aligned} \tag{2}$$

As $z_i = p^{m-i-1}$ for $i = n, ..., m-1$, we have $\sum_{i=n}^{m-1} z_i/p^{i+1} = \sum_{i=n}^{m-1} p^{m-2i-2} = (p^{m+e} - 1)/p^m(p^2-1)$. By substituting this value in (2) we obtain the result desired (1). ∎

(1.2) Lemma. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$. Set $r(G) = f_k(|G|)$ and $r(\bar{G}) = f_{\bar{k}}(|\bar{G}|)$, where $\bar{G} = G/Z(G)$. Then,*

$$\frac{z_1 - \bar{z}_1}{p^2} + \frac{z_2 - \bar{z}_2}{p^3} + \cdots + \frac{z_{n-2+e} - \bar{z}_{n-2+e}}{p^{n-1+e}} = (k - \bar{k})(p-1). \tag{3}$$

Furthermore, if $e = 0$, then $z_{n-1} = p^{n-1}$ or p^n, according as $c(G) = 0$ or $c(G) \geqslant 1$.

Proof. If $e = 1$, then $|G| = p^{2n+1}$ and $|\bar{G}| = p^{2n}$. So, by using (1.1),

$$\frac{z_0 - \bar{z}_0}{p} + \frac{z_1 - \bar{z}_1}{p^2} + \cdots + \frac{z_{n-1} - \bar{z}_{n-1}}{p^n} = (k - \bar{k})(p-1).$$

Now, $e = 1$ implies $c(G) \geqslant 1$. Hence, $z_0 = \bar{z}_0 = p$ and we obtain the equality claimed.

If $e = 0$, then $|G| = p^{2n}$ and $|\bar{G}| = p^{2n-1} = p^{2(n-1)+1}$. By (1.1), we have

$$\frac{z_0 - \bar{z}_0}{p} + \frac{z_1 - \bar{z}_1}{p^2} + \cdots + \frac{z_{n-2} - \bar{z}_{n-2}}{p^{n-1}} + \frac{z_{n-1}}{p^n} = (k - \bar{k})(p-1) + 1.$$

Now, we consider two cases. If $c(G)\geqslant 1$, then $z_0=\bar{z}_0=p$ and $[Y_{n-1}, Y_{n-1}]=[Y_{n-1}, Y_n]\leqslant Y_{2n}=1$, whence Y_{n-1} is abelian and $z_{n-1}=|s_{n-1}Y_n|=p^n$. So, $(z_0-\bar{z}_0)/p+z_{n-1}/p^n=1$ and (3) follows. If $c(G)=0$, then $G\in\mathscr{F}$ and, by [16], $z_{n-1}=\bar{z}_{n-1}=p^{n-1}$ (note that $\bar{Y}_{n-1}$ is abelian). On the other hand, $z_0=2p-1$ and $\bar{z}_0=p$. Thus, $(z_0-\bar{z}_0)/p+z_{n-1}/p^n=(p-1)/p+1/p=1$ and (3) also holds in this case. ∎

(1.3) COROLLARY. *Let G be a p-group of maximal class of order $p^m=p^{2n+e}$, $r(G)=f_k(|G|)$, and $r(\bar{G})=f_{\bar{k}}(|\bar{G}|)$, where $\bar{G}=G/Z(G)$. Then, the following inequalities hold*:

$$\bar{k}\leqslant k\leqslant p\bar{k}+n-2+e.$$

Proof. As $z_i\geqslant\bar{z}_i$ for all $i=1, ..., m-1$, (3) yields $(k-\bar{k})(p-1)\geqslant 0$, i.e., $k\geqslant\bar{k}$. On the other hand, we have $z_i\leqslant p\bar{z}_i$ for all i and, therefore,

$$\begin{aligned}(k-\bar{k})(p-1)&=\frac{z_1-\bar{z}_1}{p^2}+\frac{z_2-\bar{z}_2}{p^3}+\cdots+\frac{z_{n-2+e}-\bar{z}_{n-2+e}}{p^{n-1+e}}\\ &\leqslant(p-1)\left(\frac{\bar{z}_1}{p^2}+\cdots+\frac{\bar{z}_{n-2+e}}{p^{n-1+e}}\right)\\ &=(p-1)(n-2+e+\bar{k}(p-1)),\end{aligned}$$

where the last equality follows from (1.1) taking into account that $|\bar{G}|=p^{2(n-1+e)+1-e}$ and $\bar{z}_0=p$. Thus, $k-\bar{k}\leqslant(p-1)\bar{k}+n-2+e$ and $k\leqslant p\bar{k}+n-2+e$. ∎

Remark. In [3], P. X..Gallagher proves that, if N is a normal subgroup of a finite group G, then $r(G)\leqslant r(N)\,r(G/N)$. Consequently, $r(\bar{G})\leqslant r(G)\leqslant pr(\bar{G})$ for any p-group of maximal class G. Now, if $(s, s_1, ..., s_{m-1})$ is a generator G-system, we have $[s_{m-2}, s]=s_{m-1}$. So, $r_G(s_{m-2}Z(G))=1$ and $r(\bar{G})\leqslant r(G)<pr(\bar{G})$. Using these inequalities we could also derive the results of the last corollary. Nevertheless, the importance of Lemma (1.2) is not based on Corollary (1.3), but on the information it gives about the invariants z_i.

(1.4) COROLLARY. *Let G be a p-group of maximal class of order $p^m=p^{2n+e}$. Set $r(G)=f_k(|G|)$ and $r(\bar{G})=f_{\bar{k}}(|\bar{G}|)$. Then, $k=\bar{k}$ if and only if $z_i=\bar{z}_i$ for $i=1, ..., n-2+e$.*

Proof. This result is a direct consequence of (1.2). ∎

(1.5) COROLLARY. *Let G be a p-group of maximal class. If $c(G)=0$, then $k=\bar{k}$.*

Proof. This is immediate from (1.4) and from (2.9) of [16]. ∎

DEFINITION. Over the set $P=\{p \mid p \text{ is a prime number}\}$, we define the following functions:

If $\alpha \in \mathbb{Z}$,

$$F_\alpha(p)=\begin{cases}\dfrac{p^{\alpha+1}-1}{p-1}=p^\alpha+p^{\alpha-1}+\cdots+p+1, & \text{if } \alpha \geqslant 0;\\ 0, & \text{if } \alpha<0.\end{cases}$$

If $\alpha \in \mathbb{Z}$ is even,

$$I_\alpha(p)=\begin{cases}\dfrac{p^{\alpha+2}-1}{p^2-1}=p^\alpha+p^{\alpha-2}+\cdots+p^2+1, & \text{if } \alpha \geqslant 0;\\ 0, & \text{if } \alpha<0.\end{cases}$$

If $\alpha \in \mathbb{Z}$ is odd,

$$J_\alpha(p)=\begin{cases}\dfrac{I_{\alpha+1}(p)-(\alpha+3)/2}{p-1}=(p+1)\displaystyle\sum_{i=1}^{(\alpha+1)/2} ip^{\alpha+1-2i}, & \text{if } \alpha \geqslant 0;\\ 0, & \text{if } \alpha<0.\end{cases}$$

We note that, for $\alpha \geqslant 0$, all the three functions $F_\alpha(p)$, $I_\alpha(p)$, and $J_\alpha(p)$ are polynomials in p of degree α. Moreover, the following relations are immediate:

(i) If $\alpha \geqslant 0$, $pF_{\alpha-1}(p)=F_\alpha(p)-1$.

(ii) If $\alpha \geqslant 0$ is even, $p^2 I_{\alpha-2}(p)=I_\alpha(p)-1$.

(iii) If $\alpha \geqslant 1$ is odd, $p^2 J_{\alpha-2}(p)=J_\alpha(p)-((\alpha+1)/2)(p+1)$.

Remark. If G is a p-group of maximal class of order $p^m=p^{2n+e}$ such that $Y_1(G)$ is abelian, it is shown in [16] that $r(G)=p^{m-2}+p^2-1$, whence we deduce that

$$k=k(G)=\begin{cases}I_{m-5}(p)+J_{m-6}(p), & \text{if } m \text{ is odd};\\ J_{m-5}(p), & \text{if } m \text{ is even}.\end{cases}$$

This suggests a relation between the invariant $k(G)$ and the functions defined above. Precisely, we have the following result:

(1.6) THEOREM. *Let G be a p-group of maximal class of order p^m with $m=2n+e$ and $e\in\{0,1\}$. Suppose that $r(G)=f_k(|G|)$ and set $c=c(G)$. Then,*

(1) *If c is odd,*

$$k \geqslant \left(n - \frac{c+1}{2}\right) F_{c-2}(p) + eI_{c-3}(p) + J_{c-4}(p). \tag{4}$$

(2) *If c is even,*

$$k \geqslant \left(n - \frac{c+2}{2} + e\right) F_{c-2}(p) + eI_{c-2-2e}(p) + J_{c-3-2e}(p). \tag{5}$$

Proof. For $c=0$, the result is obvious, since the expression on the right of (5) then takes the zero value. Therefore, we can suppose in the rest of the proof that $c \geqslant 1$.

By the definition of the degree of commutativity c, $[Y_i, Y_{m-i-c}] = 1$ for all $i = 1, ..., m-c-1$. If $Y_{m-i-c} \leqslant Y_{i+1}$, then we have $z_i \geqslant |Y_{m-i-c}| = p^{i+c}$, since z_i is the number of Y_{i+1}-conjugacy classes fixed by $s_i \in Y_i - Y_{i+1}$. On the other hand, if $Y_{m-i-c} \nleqslant Y_{i+1}$, then $Y_i \leqslant Y_{m-i-c}$ and Y_i is abelian, whence $z_i = r_{Y_i}(s_i Y_{i+1}) = |s_i Y_{i+1}| = p^{m-i-1}$.

We note that the condition $Y_{m-i-c} \leqslant Y_{i+1}$ is equivalent to $m-i-c \geqslant i+1$, i.e., $2i \leqslant m-c-1 = 2n+e-c-1$. We are going to determine for what values of i the preceding inequality holds by considering the two cases c odd and c even.

(1) c is odd. If $e=0$, then $2i \leqslant 2n+e-c-1$ if and only if $i \leqslant n-(c+1)/2$. If $e=1$, $2i \leqslant 2n+e-c-1 = 2n-c$ is equivalent to $2i \leqslant 2n-c-1$, since $2i$ is even and $2n-c$ is odd. This latter inequality holds if and only if $i \leqslant n-(c+1)/2$. Hence, when c is odd, $Y_{m-i-c} \leqslant Y_{i+1}$ and $i \leqslant n-(c+1)/2$ are equivalent.

(2) c is even. If $e=0$, $2i \leqslant 2n+e-c-1 = 2n-c-1$ is equivalent to $2i \leqslant 2n-c-2$, since $2i$ is even and $2n-c-1$ is odd, and the latter inequality holds if and only if $i \leqslant n-(c+2)/2$. If $e=1$, then $2i \leqslant 2n+e-c-1 = 2n-c$ if and only if $i \leqslant n-c/2 = n-(c+2)/2+1$. So, in any case, we have $Y_{m-i-c} \leqslant Y_{i+1}$ only for $i \leqslant n-(c+2)/2+e$.

Now, we proceed to find the lower bounds of the theorem. Set $\lambda = n-(c+1)/2$ or $n-(c+2)/2+e$, according as c is odd or even. Then, we have $z_i \geqslant p^{i+c}$ for $i=1, ..., \lambda$ and $z_i = p^{m-i-1}$ for $i = \lambda+1, ..., n-1$. Bearing in mind (1), we deduce that

$$\begin{aligned} n + k(p-1) &= \frac{z_0}{p} + \frac{z_1}{p^2} + \cdots + \frac{z_{n-1}}{p^n} \\ &\geqslant 1 + p^{c-1}\lambda + \sum_{i=\lambda+1}^{n-1} p^{m-2i-2}, \end{aligned}$$

since $c \geqslant 1$ implies $z_0 = p$. As $\sum_{i=\lambda+1}^{n-1} p^{m-2i-2} = p^{m-2} \sum_{i=\lambda+1}^{n-1} p^{-2i} = p^m(p^{2(n-\lambda-1)}-1)/p^{2n}(p^2-1) = p^e(p^{2(n-\lambda-1)}-1)/(p^2-1) = p^e I_{2(n-\lambda-2)}(p)$, we get that

$$n + k(p-1) \geqslant 1 + p^{c-1}\lambda + p^e I_{2(n-\lambda-2)}(p),$$

whence

$$\begin{aligned} k(p-1) &\geqslant (p^{c-1}-1)\lambda - (n-\lambda-1) + p^e I_{2(n-\lambda-2)}(p) \\ &= (p^{c-1}-1)\lambda + e(p-1)\, I_{2(n-\lambda-2)}(p) + I_{2(n-\lambda-2)}(p) - (n-\lambda-1) \end{aligned}$$

and, finally,

$$k \geqslant \lambda F_{c-2}(p) + e I_{2(n-\lambda-2)}(p) + J_{2(n-\lambda-2)-1}(p).$$

Substituting λ for its corresponding values into the last inequality, we obtain (4) and (5). ∎

(1.7) Corollary. *Let G be a p-group of maximal class with $r(G) = f_k(|G|)$. If $c = c(G)$, then*

(1) *If Y_1 is non-abelian, $k \geqslant F_{c-2}(p)$.*

(2) *If Y_1 is abelian, $k \geqslant F_{c-3}(p)$.*

Proof. (1) If $|G| = p^m = p^{2n+e}$, the condition Y_1 non-abelian is equivalent to $c \leqslant m-4 = 2n-4+e$ and, consequently, both $n-(c+1)/2 \geqslant 1$ and $n-(c+2)/2+e \geqslant 1$ hold. Hence, we deduce from (1.6) that $k \geqslant F_{c-2}(p)$.

(2) If Y_1 is abelian, the result claimed follows from the remark made before (1.6), taking into account that $c = m-2$ and that $I_{c-3}(p) + J_{c-4}(p) \geqslant F_{c-3}(p)$ and $J_{c-3}(p) \geqslant F_{c-3}(p)$. ∎

(1.8) Corollary. *Let G be a p-group of maximal class with Y_1 non-abelian and $k = k(G) > 0$. Then,*

$$c(G) \leqslant \log_p k + 2$$

and the inequality is strict whenever $k > 1$.

Proof. If $c(G) \leqslant 2$, the above-mentioned inequality is obvious, since $\log_p k = 0$ if $k = 1$ and $\log_p k > 0$ if $k > 1$. On the other hand, if $c = c(G) \geqslant 3$, then $F_{c-2}(p) > p^{c-2}$ and (1.7) gives $k > p^{c-2}$, whence $c < \log_p k + 2$. ∎

The last corollary is an improvement, when Y_1 is non-abelian, of the result $c(G) \leqslant \log_p k + 3$, which is proved in [16].

A natural question that arises after Theorem (1.6) is the following one: For what family of p-groups of maximal class does $k(G)$ coincide with the lower bound given in that theorem? That is, if $m=2n+e\geqslant 4$ and we define for $c\in\{0,1,...,m-4\}\cup\{m-2\}$

$$B_{m,p}(c) = \begin{cases} \left(n-\dfrac{c+1}{2}\right)F_{c-2}(p)+eI_{c-3}(p)+J_{c-4}(p), & \text{if } c \text{ is odd,} \\ \left(n-\dfrac{c+2}{2}+e\right)F_{c-2}(p)+eI_{c-2-2e}(p)+J_{c-3-2e}(p), & \text{if } c \text{ is even,} \end{cases}$$

when does the equality $k(G)=B_{m,p}(c(G))$ hold? Having a look at the proof of (1.6), it is clear that, if $c(G)\geqslant 1$, the equality holds if and only if $z_i=p^{i+c(G)}$ whenever $0<2i\leqslant m-c(G)-1$ (this is not true for $c(G)=0$; for those groups, we determine in the next section when we have $k(G)=B_{m,p}(c(G))=0$). This local information will be transformed into global conditions on the commutator subgroups $[Y_i, Y_j]$ of the group G. To do this, we need the following lemmas.

(1.9) LEMMA. *Let G be a p-group of maximal class of order p^m. If $i, j\geqslant 1$ and $i+j=m-1$, then*

$$[Y_i, Y_j]=\begin{cases} 1, & \text{if } c(G)\geqslant 1; \\ Y_{m-1}, & \text{if } c(G)=0. \end{cases}$$

Proof. If $c(G)\geqslant 1$, then $[Y_i, Y_j]\leqslant Y_{i+j+c(G)}\leqslant Y_m=1$. If $c(G)=0$, we know that $Y_1\neq C_G(Y_{m-2})$ (cf. [1]). So, $1\neq[Y_1, Y_{m-2}]\leqslant Y_{m-1}$ and necessarily $[Y_1, Y_{m-2}]=Y_{m-1}$. We prove by induction on i that $[Y_i, Y_{m-i-1}]=Y_{m-1}$ for $i=1,...,m-2$. We have just seen that the result is true for $i=1$. We suppose it true for i $(1\leqslant i\leqslant m-3)$ and prove it for $i+1$. Set $[Y_i, Y_{m-i-2}]=Y_u$. It it were $u=m-2$, we would have $c(G/Y_{m-1})=0$, which is impossible according to [1]. Hence, $Y_u\leqslant Y_{m-1}=Z(G)$. Now, since $[Y_{i+1}, Y_{m-i-2}]\in\{1, Y_{m-1}\}$, P. Hall's Three Subgroup Lemma applied to the triple commutators

$$[Y_i, Y_{m-i-2}, G]=[Y_u, G]=1,$$
$$[Y_{m-i-2}, G, Y_i]=[Y_{m-i-1}, Y_i]=Y_{m-1},$$
$$[G, Y_i, Y_{m-i-2}]=[Y_{i+1}, Y_{m-i-2}],$$

yields $[Y_{i+1}, Y_{m-i-2}]=Y_{m-1}$, which completes the proof. ∎

(1.10) LEMMA. *Let G be a p-group of maximal class of order p^m and $\bar{G}=G/Z(G)$. Suppose $i\in\{1,...,m-c(G)-2\}$ and $[Y_i, Y_{m-i-c(G)-1}]\neq 1$. Then, the following assertions hold:*

(1) *$r_G(Y_i - Y_{i+1}) = r_{\bar{G}}(\bar{Y}_i - \bar{Y}_{i+1})$ and $|Cl_G(x)| = p\,|Cl_{\bar{G}}(\bar{x})|$ for every $x \in Y_i - Y_{i+1}$.*

(2) *If $i \leqslant m - i - c(G) - 1$, then $z_i = \bar{z}_i$.*

Proof. As $[Y_i, Y_{m-i-c(G)-1}] \neq 1$, necessarily $[Y_i, Y_{m-i-c(G)-1}] = Y_{m-1}$. On the other hand, $[Y_{i+1}, Y_{m-i-c(G)-1}] = 1$. Consequently, for any $y \in Y_i - Y_{i+1}$ there exists $g \in Y_{m-i-c(G)-1}$ such that $[y, g] \in Y_{m-1} - \{1\}$, i.e., $y^g = yz$ with $z \in Y_{m-1} - \{1\}$. Hence, $y^{g^j} = yz^j$ for all $j \in \{0, ..., p-1\}$. We deduce that $r_G(yZ(G)) = 1$ and, if $i \leqslant m - i - c(G) - 1$, that $r_{Y_i}(yZ(G)) = 1$. Now, if $\{\bar{y}_1, ..., \bar{y}_r\}$ is a complete set of representatives of the $\bar{G}$-conjugacy classes that make up $\bar{Y}_i - \bar{Y}_{i+1}$, we have $r_G(Y_i - Y_{i+1}) = \sum_{j=1}^r r_G(y_j Z(G)) = r = r_{\bar{G}}(\bar{Y}_i - \bar{Y}_{i+1})$. Arguing in the same way we obtain $z_i = \bar{z}_i$ when $i \leqslant m - i - c(G) - 1$. ∎

(1.11) Corollary. *Let G be a p-group of maximal class of order p^m with $c(G) = 0$. Then,*

(1) *If $i \in \{1, ..., m-2\}$, $r_G(Y_i - Y_{i+1}) = r_{\bar{G}}(\bar{Y}_i - \bar{Y}_{i+1})$ and $|Cl_G(x)| = p\,|Cl_{\bar{G}}(\bar{x})|$ for every $x \in Y_i - Y_{i+1}$.*

(2) *If $i \in \{1, ..., n-1\}$, then $z_i = \bar{z}_i$.*

Proof. It is immediate from (1.9) and (1.10). ∎

(1.12) Lemma. *Let G be a p-group of maximal class with $c(G) \geqslant 1$ and $\bar{G} = G/Z(G)$. Suppose that $k(G) = B_{m,p}(c(G))$. Then, $k(\bar{G}) = B_{m-1,p}(c(\bar{G}))$ and*

(1) *If $c(G) \geqslant m - 4$, then $Y_1(\bar{G})$ is abelian and $c(\bar{G}) = m - 3$.*

(2) *If $c(G) \leqslant m - 5$, then $c(\bar{G}) = c(G)$.*

Proof. (1) If $c(G) \geqslant m - 4$, then $[Y_1, Y_1] = [Y_1, Y_2] \leqslant Y_{m-1}$ and $Y_1(\bar{G})$ is abelian. Hence, $c(\bar{G}) = (m-1) - 2 = m - 3$. It follows from the remark previous to (1.6) that $k(\bar{G}) = B_{m-1,p}(c(\bar{G}))$, just taking into account that $c(\bar{G})$ will be odd or even at the same time as $m - 1$.

(2) Suppose now that $c(G) \leqslant m - 5$. Setting $c = c(G)$, we have $c(\bar{G}) \geqslant c$. As $k(G) = B_{m,p}(c)$, if $0 < 2i \leqslant m - c - 1$, then $z_i = p^{i+c}$, whence $\bar{z}_i \leqslant p^{i+c}$. In particular, $\bar{z}_1 \leqslant p^{c+1}$ and necessarily $[\bar{Y}_1, \bar{Y}_{(m-1)-c-2}] \neq \bar{1}$ (note that $(m-1) - c - 2 \geqslant 2$, since $c \leqslant m - 5$). Consequently, $c(\bar{G}) = c$ and $\bar{z}_i \geqslant p^{i+c}$ for $0 < 2i \leqslant (m-1) - c - 1$. Thus, we obtain $\bar{z}_i = p^{i+c}$ whenever $0 < 2i \leqslant (m-1) - c - 1$, whence $k(\bar{G}) = B_{m-1,p}(c(\bar{G}))$. ∎

(1.13) Theorem. *Let G be a p-group of maximal class of order p^m such that $c = c(G) \geqslant 1$. Then, the following assertions are equivalent:*

(1) $k(G) = B_{m,p}(c)$.

(2) $z_i = p^{i+c}$ *whenever* $0 < 2i \leqslant m - c - 1$.

(3) $[Y_i, Y_j] = Y_{i+j+c}$ *whenever* $i \neq j$.

Proof. The equivalence between (1) and (2) has already been mentioned.

Now, we suppose (1) and (2) true and prove (3). We use induction on $|G|$. If $|G| = p^4$, the result is obvious. Assume that (3) holds for p-groups of maximal class of order smaller than $|G|$. If $c \geqslant m-4$, (3) is evident (just note that, when $c = m-4$, $i+j+c < m$ and $i \neq j$ imply $\{i, j\} = \{1, 2\}$). If $c \leqslant m-5$, (1.12) yields $c(\bar{G}) = c$ and $k(\bar{G}) = B_{m-1,p}(c(\bar{G}))$. From the inductive hypothesis, it follows that $[\bar{Y}_i, \bar{Y}_j] = \bar{Y}_{i+j+c}$ whenever $i \neq j$. We deduce that $[Y_i, Y_j] = Y_{i+j+c}$ if $i+j+c \neq m-1$ and $i \neq j$. If $i+j+c = m-1$, we can suppose without loss of generality that $i < j = m-i-c-1$, i.e., $Y_{m-i-c-1} \leqslant Y_{i+1}$. If it were $[Y_i, Y_{m-i-c-1}] = 1$, we would have $z_i \geqslant |Y_{m-i-c-1}| = p^{i+c+1}$, a contradiction with (2). Hence, $[Y_i, Y_{m-i-c-1}] = Y_{m-1}$ and the result follows.

Finally, we derive (2) from (3). We also argue by induction on $|G|$. If $c \geqslant m-4$, the result is evident (in particular, when $|G| = p^4$). So, we can suppose $c \leqslant m-5$ and that the result is true for groups of smaller order. Then, clearly (3) implies $c(\bar{G}) = c$ and $[\bar{Y}_i, \bar{Y}_j] = \bar{Y}_{i+j+c}$ when $i \neq j$. Take firstly i such that $0 < 2i \leqslant m - c - 2$. The inductive hypothesis yields $\bar{z}_i = p^{i+c}$. If $z_i > \bar{z}_i$, then (1.10) gives $[Y_i, Y_{m-i-c-1}] = 1$, impossible, since $i < m-i-c-1$. Hence, $z_i = p^{i+c}$ when $0 < 2i \leqslant m-c-2$. If $2i = m-c-1$, then $[Y_i, Y_i] = [Y_i, Y_{i+1}] = [Y_i, Y_{m-i-c}] = 1$. Therefore, Y_i is abelian and $z_i = r_{Y_i}(s_i Y_{i+1}) = |s_i Y_{i+1}| = p^{m-i-1} = p^{i+c}$. ∎

Example. The existence of p-groups of maximal class satisfying the equality $k(G) = B_{m,p}(c(G))$ is assured in [13]. In that paper, for $m \leqslant p$ given and $c \in \{1, ..., m-4\} \cup \{m-2\}$, $c \leqslant p-m+2$, B. A. Panferov constructs the following Lie algebra L of maximal nilpotency class and dimension m over $\mathbb{F}_p$: given a basis $\{e_0, ..., e_{m-1}\}$, the Lie product in L is determined by the products

$$[e_i, e_0] = -[e_0, e_i] = (i+c-1)e_{i+1}, \qquad i = 1, 2, ..., m-2,$$
$$[e_0, e_0] = [e_{m-1}, e_0] = [e_0, e_{m-1}] = 0,$$

and, if $i, j \geqslant 1$,

$$[e_i, e_j] = \begin{cases} (i-j)e_{i+j+c}, & \text{if } \; i+j+c \leqslant m-1; \\ 0, & \text{if } \; i+j+c > m-1. \end{cases}$$

As $m \leqslant p$, by means of the Hausdorff formula (cf. [7]), one obtains a p-group of maximal class G of degree of commutativity $c(G) = c$ verifying

$[Y_i, Y_j] = Y_{i+j+c}$ whenever $i \neq j$. Now, taking into account (1.13), it follows that $k(G) = B_{m,p}(c(G))$.

We note that, in [6], L. G. Kovács and C. R. Leedham-Green also consider such a Lie algebra for the particular case $c = 1$. The interest of their method lies in the fact that they present it as an algebra of $(m+1)$-by-$(m+1)$ matrices over the field $\mathbb{F}_p$. As a consequence, the p-group G of maximal class they obtain by means of the Hausdorff formula is a subgroup of the multiplicative group of the upper unitriangular matrices. Here, following their ideas, we generalize that example to the case when $c \in \{1, ..., m-4\} \cup \{m-2\}$ and $c \leqslant p-m+1$ (supposed $m \leqslant p$).

First, we treat the case $c \leqslant p-m$. In the associative algebra of the $(m+c)$-by-$(m+c)$ matrices over $\mathbb{F}_p$, we consider the Lie product $[x, y] = xy - yx$. Then, defining

$$u_i = \sum_{k=1}^{m+c-i} ke(k, k+i) \qquad \text{for} \quad i = 1, ..., m+c-1,$$

where $e(i, j)$ is the matrix with the (i, j) entry equal to 1 and the rest equal to zero, we have

$$[u_i, u_j] = \begin{cases} (i-j)u_{i+j}, & \text{if} \quad i+j \leqslant m+c-1; \\ 0, & \text{if} \quad i+j > m+c-1. \end{cases}$$

Setting $e_0 = u_1$ and $e_i = u_{i+c}$ for $i = 1, ..., m-1$, we get for $\{e_0, ..., e_{m-1}\}$ the same Lie product relations as those given by Panferov. Now, we put $L = \bigoplus_{i=0}^{m-1} \mathbb{F}_p e_i$. L is a Lie subalgebra of the associative algebra T of nilpotent upper triangular matrices. As $T^{m+c} = 0$ and $c \leqslant p-m$, it follows that $T^p = 0$ and $\exp x = \sum_{j=0}^{\infty} x^j/j! = 1 + \sum_{j=1}^{p-1} x^j/j! \in 1 + T$ is well-defined for any $x \in T$. If we set $G = \{\exp x \mid x \in L\}$, the Hausdorff formula

$$\exp x \exp y = \exp \Phi(x, y),$$

where

$$\Phi(x, y) = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}[[x, y], y] - \tfrac{1}{12}[[x, y], x] + \cdots,$$

yields that G is a subgroup of the multiplicative group $1 + T$ of the upper unitriangular matrices. Now, if we define $g_i = \exp e_i$ for $i = 0, ..., m-1$, reasoning as Kovács and Leedham-Green it follows that the subgroups of G of the form $\langle g_i, ..., g_{m-1} \rangle$ with $i \in \{0, ..., m-1\}$ are pairwise distinct. Hence, $|G| = p^m$. We also have that

$$g_0^p = \cdots = g_{m-1}^p = 1 \tag{6}$$

and that there exist integers $\gamma(i, j, k)$ such that

$$[g_i, g_0] = \begin{cases} g_{i+1}^{i+c-1} \prod_{k=i+2}^{m-1} g_k^{\gamma(i,0,k)}, & \text{if } 1 \leqslant i \leqslant m-2; \\ 1, & \text{if } i = m-1, \end{cases} \tag{7}$$

and, if $i \geqslant j \geqslant 1$,

$$[g_i, g_j] = \begin{cases} g_{i+j+c}^{i-j} \prod_{k=i+2j+2c}^{m-1} g_k^{\gamma(i,j,k)}, & \text{if } i+j+c \leqslant m-1; \\ 1, & \text{if } i+j+c > m-1. \end{cases} \tag{8}$$

From these formulas, one can easily deduce that G is a p-group of maximal class satisfying $Y_i(G) = \langle g_i, ..., g_{m-1} \rangle$ for each $i \in \{1, ..., m-1\}$ and $[Y_i(G), Y_j(G)] = Y_{i+j+c}(G)$ whenever $i \neq j$.

Finally, when $c = p - m + 1$, we consider the associative subring R of T generated by L. Then, we have $R^p = 0$ and, from this point on, we argue as in the case $c \leqslant p - m$.

As an application of this procedure, we consider the particular case $m = 8$, for which it is not difficult to find systems of integers $\{\gamma(i, j, k)\}$ such that (6), (7), and (8) define a p-group of maximal class of order p^8 and degree of commutativity $c \in \{2, 3, 4, 6\}$ (supposed $p \geqslant c + 7$). For instance, we have the family of groups defined by the relations (6) and

$$[g_i, g_0] = \begin{cases} g_{i+1}^{i+c-1} g_7^{\lambda_i}, & \text{if } 1 \leqslant i \leqslant 6; \\ 1, & \text{if } i = 7, \end{cases}$$

and, if $i \geqslant j \geqslant 1$,

$$[g_i, g_j] = \begin{cases} g_{i+j+c}^{i-j}, & \text{if } i+j+c \leqslant 7; \\ 1, & \text{if } i+j+c > 7, \end{cases}$$

where λ_i is arbitrary for $i \in \{1, 2, 3, 4\}$, $\lambda_5 \equiv 6 \pmod{p}$ if $c = 2$ and is arbitrary if $c \geqslant 3$, and $\lambda_6 \equiv 0 \pmod{p}$.

(1.14) COROLLARY. *Let G be a p-group of maximal class such that $c(G) \geqslant 1$. If $k(G) = B_{m,p}(c(G))$, then $k(H) = B_{m-1,p}(c(H))$ for any $H \in \mathcal{M}(G) - \{Y_1\}$. Furthermore,*

(1) *If $c(G) \geqslant m-5$, then $Y_1(H)$ is abelian and $c(H) = m-3$.*

(2) *If $c(G) \leqslant m-6$, then $c(H) = c(G) + 1$.*

Proof. (1) If $c(G) \geqslant m-5$, then $[Y_2, Y_2] = [Y_2, Y_3] \leqslant Y_m = 1$ and $Y_1(H) = Y_2$ is abelian. Hence, $c(H) = (m-1) - 2 = m-3$ and $k(H) = B_{m-1,p}(c(H))$.

(2) Now, suppose that $c = c(G) \leqslant m-6$. We have $c(H) \geqslant c+1$ (cf. [16, (1.2)]). As $k(G) = B_{m,p}(c(G))$, (1.13) yields $[Y_i, Y_j] = Y_{i+j+c}$ for $i \neq j$, whence $[Y_i(H), Y_j(H)] = Y_{i+j+c+1}(H)$. Now, $c \leqslant m-6$ implies $c(H) = c+1$ and, using (1.13) again, we obtain $k(H) = B_{m-1,p}(c(H))$. ∎

(1.15) THEOREM. *Let G be a p-group of maximal class of order p^m and $r(G) = f_k(|G|)$. If $m \geqslant 2k+5$, then $c(G) \leqslant 1$.*

Proof. We suppose that $c(G) \geqslant 2$ and will reach the inequality $k(G) < B_{m,p}(c(G))$, which is a contradiction with (1.6). We are going to do this in two steps:

(i) If $c \geqslant 2$, then $B_{m,p}(c) \geqslant B_{m,p}(2)$.

(ii) If $m \geqslant 2k+5$, then $B_{m,p}(2) > k$.

(i) The range of the function $B_{m,p}(c)$ is $\{0, 1, ..., m-4\} \cup \{m-2\}$. Let's first consider the case in which $c \neq m-2$. If c is odd, we have

$$\begin{aligned} B_{m,p}(c) - B_{m,p}(2) &\geqslant \left(n - \frac{c+1}{2}\right) F_{c-2}(p) - (n-2+e) \\ &= \left(n - \frac{c+1}{2}\right)(p^{c-2} + \cdots + p) + 2 - e - \frac{c+1}{2} \\ &\geqslant (c-2) + 2 - e - \frac{c+1}{2} \geqslant \frac{c-3}{2} \geqslant 0, \end{aligned}$$

where we have used that $n - (c+1)/2 \geqslant 1$ (since $c \leqslant m-4$), that $p^{c-2} + \cdots + p \geqslant c-2$ and that $c \geqslant 3$, since c is odd and $c \geqslant 2$ by hypothesis. If c is even, the reasoning is completely similar.

In the case $c = m-2$, the remark before (1.6) yields $B_{m,p}(m-2) \geqslant p^{2n+e-5}$. Now, from the inequality $p^x \geqslant 1 + x \ln p \geqslant 1 + x/2$ for $x \geqslant 0$, we deduce that $B_{m,p}(m-2) \geqslant 1 + (2n+e-5)/2 \geqslant 1 + (2n-6+2e)/2 = n-2+e = B_{m,p}(2)$.

(ii) If $m = 2n+e \geqslant 2k+5$, then $2n+2e \geqslant 2k+6$, since $2n+2e$ is even and $2k+5$ is odd. So, we have $n+e \geqslant k+3$ and, consequently, $B_{m,p}(2) = n-2+e \geqslant k+1 > k$, as we wanted to prove. ∎

DEFINITION. Let G be a p-group of maximal class. We say that a series of subgroups of G,

$$G = H_0 > H_1 > H_2 > \cdots > H_s$$

is a *$\mathcal{M}$-chain of G of length s* when the two following conditions hold:

(1) $s \leqslant m-3$.

(2) $H_1 \in \mathcal{M}(G) - \{Y_1(G), C_G(Y_{m-2}(G))\}$, and $H_i \in \mathcal{M}(H_{i-1}) - \{Y_1(H_{i-1})\}$ for $2 \leqslant i \leqslant s$.

It is clear from the above definition (cf. [1, (3.1); 16, (1.2)]) that every H_i is a p-group of maximal class of order p^{m-i} and that $Y_j(H_i) = Y_{i+j}(G)$ for $1 \leqslant i \leqslant \min\{s, m-4\}$ and $j \geqslant 1$ (note that, if $s = m-3$, H_{m-3} is a p-group of maximal class of order p^3 and $Y_1(H_{m-3})$ makes no sense).

(1.16) LEMMA. *Let G be a p-group of maximal class. If $x, y \in Y_1 - Y_2$ and $u \geqslant 0$, then $x^{p^u} \in Y_v$ if and only if $y^{p^u} \in Y_v$.*

Proof. We use induction on u. If $u=0$, the result is trivial, since $x, y \in Y_1 - Y_2$. If $u=1$, we distinguish two cases. If $m > p+1$, it is well known that $x^p, y^p \in Y_p - Y_{p+1}$ (cf. [1]) and the result is true. If $m \leqslant p+1$, we have $\exp Y_2 = p$ (again, cf. [1]). Moreover, as $x, y \in Y_1 - Y_2$, there exist $j \in \{1, ..., p-1\}$ and $z \in Y_2$ such that $x = y^j z$. Since Y_1 is a regular p-group, there exists $z' \in \mho_1(\langle y^j, z\rangle')$ such that $x^p = (y^j z)^p = y^{jp} z^p z'$. Now, taking into account that $z^p = 1$, since $z \in Y_2$ and that $\mho_1(\langle y^j, z\rangle') \leqslant \mho_1(Y_1') \leqslant \mho_1(Y_2) = 1$, we deduce that $x^p = (y^p)^j$ with $j \in \{1, ..., p-1\}$ and, consequently, $x^p \in Y_v$ if and only if $y^p \in Y_v$.

Now, we assume the result true for $u \geqslant 1$ and prove it for $u+1$. If $x^{p^u}, y^{p^u} \in Y_{m-2}$, the result trivially holds, since $\exp Y_{m-2} = p$. Otherwise, from the inductive hypothesis, there exists $w \in \{1, ..., m-3\}$ such that $x^{p^u}, y^{p^u} \in Y_w - Y_{w+1}$. Then, we consider a $\mathcal{M}$-chain of G, $G = H_0 > H_1 > \cdots > H_{m-4}$. We have $Y_w - Y_{w+1} = Y_1(H_{w-1}) - Y_2(H_{w-1})$ and, the result being true in H_{w-1} for $u=1$, we deduce that $x^{p^{u+1}} = (x^{p^u})^p \in Y_{v-w+1}(H_{w-1})$ if and only if $y^{p^{u+1}} = (y^{p^u})^p \in Y_{v-w+1}(H_{w-1})$, that is, $x^{p^{u+1}} \in Y_v$ if and only if $y^{p^{u+1}} \in Y_v$, which proves the lemma. ∎

(1.17) THEOREM. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$. Set $c = c(G)$ and suppose that $r(G) = f_k(|G|)$ with $k < F_\alpha(p)$. Then, the following inequality holds:*

$$m \leqslant 2p + \alpha(p-1) - c(p-2).$$

Proof. Firstly, we prove that $z_1 < p^{\alpha+3}$. In fact, if it were $z_1 \geqslant p^{\alpha+3}$, according to (1) we would have

$$n + k(p-1) = \frac{z_0}{p} + \frac{z_1}{p^2} + \cdots + \frac{z_{n-1}}{p^n} \geqslant p^{\alpha+1} + n - 1,$$

whence $k \geqslant (p^{\alpha+1} - 1)/(p-1) = F_\alpha(p)$, which is a contradiction.

Next, we prove that $s_1^{p^{\alpha+2-c}} \in Y_{m-c-1}$. We note that $s_1^{p^{\alpha+2-c}}$ makes sense, because Corollary (1.7) yields $\alpha \geqslant c-2$. If $s_1^{p^{\alpha+2-c}} \notin Y_{m-c-1}$, then, by virtue of (1.16), $x^{p^{\alpha+2-c}} \notin Y_{m-c-1}$ for all $x \in Y_1 - Y_2$. Since $\langle x, Y_{m-c-1}\rangle \leqslant C_{Y_1}(x)$, we deduce that $|C_{Y_1}(x)| \geqslant p^{\alpha+3-c}\,|Y_{m-c-1}| = p^{\alpha+4}$ for all $x \in Y_1 - Y_2$. Consequently, $|Cl_{Y_1}(x)| \leqslant |Y_1|/p^{\alpha+4} = p^{m-\alpha-5}$ for $x \in Y_1 - Y_2$ and $z_1 = r_{Y_1}(s_1 Y_2) \geqslant |s_1 Y_2|/p^{m-\alpha-5} = p^{\alpha+3}$, which is impossible.

If we suppose $m \geqslant 2p + \alpha(p-1) - c(p-2) + 1$, then $1 + (\alpha+2-c)(p-1) < m-c-1$ and $Y_{m-c-1} < Y_{1+(\alpha+2-c)(p-1)}$. Under that assumption, we also have $m > 2 + (\alpha+2-c)(p-1)$ and, consequently, if we take a $\mathscr{M}$-chain $G = H_0 > H_1 > \cdots > H_{m-4}$, then $|H_j| > p^{p+1}$ for all $j = 0, ..., (\alpha+1-c)(p-1)$. So, if $j \in \{0, ..., (\alpha+1-c)(p-1)\}$, then $x_j^p \in Y_p(H_j) - Y_{p+1}(H_j) = Y_1(H_{j+(p-1)}) - Y_2(H_{j+(p-1)})$ for every $x_j \in Y_1(H_j) - Y_2(H_j)$. Applying repeatedly this property, we get that $s_1^{p^{\alpha+2-c}} \in Y_p(H_{(\alpha+1-c)(p-1)}) - Y_{p+1}(H_{(\alpha+1-c)(p-1)}) = Y_{1+(\alpha+2-c)(p-1)} - Y_{2+(\alpha+2-c)(p-1)}$. Hence, $s_1^{p^{\alpha+2-c}} \notin Y_{m-c-1}$, contrary to what we have proved. In short, it must be $m \leqslant 2p + \alpha(p-1) - c(p-2)$. ∎

(1.18) Corollary. *If G is a p-group of order p^m such that $r(G) = f_0(|G|)$, then $m \leqslant p+2$ (J. Poland). If G is a p-group of maximal class of order p^m and $r(G) = f_k(|G|)$ with $k \leqslant p$, then $m \leqslant 2p+1$.*

Proof. In [14], J. Poland shows that $r(G) = f_0(|G|)$ implies that G is a p-group of maximal class. If $c(G) = 0$, it is well known (cf. [1]) that $m \leqslant p+1$. If $c = c(G) \geqslant 1$, then (1.17) used with $\alpha = 0$ or 1 yields $m \leqslant 2p - c(p-2) \leqslant p+2$ or $m \leqslant 2p + (p-1) - c(p-2) \leqslant 2p+1$, according as $k(G) = 0$ or $k(G) \leqslant p$. ∎

(1.19) Corollary. *Let G be a p-group of maximal class of order p^m. If $r(G) = f_k(|G|)$ and $k < F_\alpha(p)$, then $c(G) \leqslant 2 + \alpha(p-1)/(p-2)$.*

Proof. It is immediate from Theorem (1.17), since $m \geqslant 4$ implies $4 \leqslant 2p + \alpha(p-1) - c(G)(p-2)$. ∎

Remark. This last result can be compared with Corollary (1.8).

The following two lemmas will be useful in the next sections when, assumed that $k(G) \leqslant 1$, we try to determine the values of the invariant k, $k(H_i)$, and $k(Y_i)$, for the subgroups of a $\mathscr{M}$-chain of G and of its extended lower central series.

(1.20) Lemma. *Let G be a p-group of maximal class of order p^m. Consider a $\mathscr{M}$-chain of G, $G = H_0 > H_1 > \cdots > H_{m-3}$. Set $|H_i| = p^{2n_i+e_i}$*

with $e_i \in \{0, 1\}$ *for* $i = 0, ..., m-3$. *Take* $\alpha_i = 0$ *for* $i \geqslant 1$ *and* $\alpha_0 = 0$ *or* 1, *according as* $c(G) \geqslant 1$ *or* $c(G) = 0$. *Then, the equality*

$$z_i = p\{1 + (p-1)(pk(H_{i-1}) - k(H_i) + n_{i-1} - 1 - \alpha_{i-1})\}$$

holds for $i = 1, ..., m-3$.

Proof. Let $i \in \{1, ..., m-3\}$ and $x \in H_{i-1} - Y_1(H_{i-1}) = H_{i-1} - Y_i(G)$. As $c(H_j) \geqslant 1$ for $j \geqslant 1$, we have $r_{H_{i-1}}(xY_1(H_{i-1})) = p + \alpha_{i-1}(p-1)$. Hence, [2, Note E] yields

$$\begin{aligned} p^2 r(H_{i-1}) &= p(p^2-1)(p + \alpha_{i-1}(p-1)) + pr(Y_i(G)) \\ &= p(p^2-1)(p + \alpha_{i-1}(p-1)) + (p^2-1)z_i + r(Y_{i+1}(G)). \end{aligned} \tag{9}$$

On the other hand, if $y \in H_i - Y_{i+1}(G)$, we have $r_{H_i}(yY_{i+1}(G)) = p$, whence

$$pr(H_i) = p(p^2-1) + r(Y_{i+1}(G)). \tag{10}$$

Combining (9) and (10) we obtain

$$p^2 r(H_{i-1}) = p(p^2-1)(p + \alpha_{i-1}(p-1)) + (p^2-1)z_i + pr(H_i) - p(p^2-1).$$

It follows that p divides z_i. Setting $z_i = pz'_i$, we get

$$pr(H_{i-1}) = (p^2-1)(p + \alpha_{i-1}(p-1)) + (p^2-1)z'_i + r(H_i) - (p^2-1).$$

If $|H_{i-1}| = p^{2n_{i-1} + e_{i-1}}$, then $|H_i| = p^{2(n_{i-1} + e_{i-1} - 1) + 1 - e_{i-1}}$, i.e., $n_i = n_{i-1} + e_{i-1} - 1$ and $e_i = 1 - e_{i-1}$. So,

$$\begin{aligned} &n_{i-1}\, p(p^2-1) + p^{1+e_{i-1}} + k(H_{i-1})\, p(p^2-1)(p-1) \\ &\quad = (p^2-1)(p + \alpha_{i-1}(p-1) + z'_i - 1) \\ &\qquad + (n_{i-1} + e_{i-1} - 1)(p^2-1) + p^{1-e_{i-1}} + k(H_i)(p^2-1)(p-1). \end{aligned}$$

Taking into account that $e_{i-1} \in \{0, 1\}$ implies $p^{1+e_{i-1}} - p^{1-e_{i-1}} = e_{i-1}(p^2-1)$, we have

$$\begin{aligned} &n_{i-1}\, p + e_{i-1} + k(H_{i-1})\, p(p-1) \\ &\quad = p + \alpha_{i-1}(p-1) + z'_i - 1 + n_{i-1} + e_{i-1} - 1 + k(H_i)(p-1), \end{aligned}$$

that is,

$$\begin{aligned} z'_i = n_{i-1}(p-1) &+ k(H_{i-1})\, p(p-1) - k(H_i)(p-1) \\ &- \alpha_{i-1}(p-1) - (p-1) + 1 \end{aligned}$$

and, consequently,

$$z_i = p\{1 + (p-1)(pk(H_{i-1}) - k(H_i) + n_{i-1} - 1 - \alpha_{i-1})\}. \quad \blacksquare$$

(1.21) LEMMA. *Let $G=Y_0$ be a p-group of maximal class of order p^m. Set $|Y_i|=p^{2n_i+e_i}$ with $e_i\in\{0,1\}$ for $i=0,\dots,m$. Then,*

$$z_i=1+(p-1)(pk(Y_i)-k(Y_{i+1})+n_i)$$

holds for $i=0,\dots,m-1$.

Proof. As $s_i\in Y_i-Y_{i+1}$, we have $pr(Y_i)=z_i(p^2-1)+r(Y_{i+1})$, that is,

$$\begin{aligned}&n_i p(p^2-1)+p^{1+e_i}+k(Y_i)\,p(p^2-1)(p-1)\\&\quad=z_i(p^2-1)+(n_i+e_i-1)(p^2-1)+p^{1-e_i}+k(Y_{i+1})(p^2-1)(p-1),\end{aligned}$$

since $|Y_i|=p^{2n_i+e_i}$ and $|Y_{i+1}|=p^{2(n_i+e_i-1)+1-e_i}$. Now, $p^{1+e_i}-p^{1-e_i}=e_i(p^2-1)$ yields

$$pn_i+e_i+p(p-1)\,k(Y_i)=z_i+n_i+e_i-1+(p-1)\,k(Y_{i+1}),$$

whence

$$z_i=1+(p-1)(pk(Y_i)-k(Y_{i+1})+n_i). \quad \blacksquare$$

2. p-GROUPS SATISFYING $k(G)=0$

In this section, we consider the family of the p-groups satisfying $r(G)=f_0(|G|)$. The main results about these groups are given in [14, 16]. In [14], J. Poland proves that such a group has always maximal class. Furthermore, provided that $|G|=p^m$ with $m=2n+e\geqslant 5$, then $c(G)\in\{0,1\}$ and:

(i) If $c(G)=0$, then $\nabla_G=(p,p-1,\overset{n-1}{\cdots},p-1,p^2-p,\overset{n-4}{\cdots},p^2-p,2(p^2-p),(p-1)^2)$ if $n\geqslant 4$ or $\nabla_G=(p,p-1,p-1,2p^2-p-1,(p-1)^2)$ if $n=3$.

(ii) If $c(G)=1$, then $\nabla_G=(p,p-1,\overset{n-2+e}{\cdots},p-1,p^2-1,p^2-p,\overset{n-1}{\cdots},p^2-p)$.

In [16], A. Vera-López and B. Larrea find the values of the numerical G-systems of such groups. Namely, they prove that, if $m\geqslant 5$:

(i) If $e=1$, then $\sigma_G=(p,p^2,p^3,\dots,p^{n-1},p^n,\tau_n)$.

(ii) If $e=0$, then

$$\sigma_G=\begin{cases}(p,p^2,p^3,\dots,p^{n-1},\tau_n), & \text{if } c(G)=1;\\(2p-1,p^2,p^3,\dots,p^{n-1},p^{n-1},\tau_{n-1}), & \text{if } c(G)=0.\end{cases}$$

And, if $m=4$, $\sigma_G=(p, p^2, p, 1)$. It follows that $z_i=p^{i+1}$ for $i=1, ..., n-2$ and that $z_{n-1}=p^{n-1}$ or p^n, according as $c(G)=0$ or 1, which we will use later on. We are going to complete this information by characterizing these groups in terms of the commutator subgroups $[Y_i, Y_j]$ and determining the values of k, $k(H_i)$ and $k(Y_i)$, of the subgroups of a $\mathcal{M}$-chain of G and of the extended lower central series of G, respectively.

(2.1) THEOREM. *Let G be a p-group of maximal class of order p^m. Then, $r(G)=f_0(|G|)$ if and only if $[Y_i, Y_j]=Y_{i+j+1}$ whenever $i\neq j$ and $i+j\leqslant m-2$.*

Proof. First, we suppose $r(G)=f_0(|G|)$. If $|G|=p^4$, the result is obvious. If $|G|\geqslant p^5$, then $c(G)\in\{0, 1\}$. Since $k(G)=0$, it follows that $k(G)=B_{m,p}(c(G))$. Hence, if $c(G)=1$, Theorem (1.13) yields $[Y_i, Y_j]=Y_{i+j+1}$ whenever $i\neq j$. If $c(G)=0$, then $|G|\geqslant p^6$. Consequently, $|\bar{G}|\geqslant p^5$, and $k(\bar{G})=0$ implies $c(\bar{G})=1$. So, $[\bar{Y}_i, \bar{Y}_j]=\bar{Y}_{i+j+1}$ for $i\neq j$, whence $[Y_i, Y_j]=Y_{i+j+1}$ whenever $i\neq j$ and $i+j\leqslant m-3$. Finally, if $i+j=m-2$, that is, if $j=m-i-2$, then $Y_{m-1}=[Y_i, Y_{m-i-1}]\leqslant[Y_i, Y_j]$ implies $[Y_i, Y_j]=Y_{m-1}=Y_{i+j+1}$.

Now, we assume $[Y_i, Y_j]=Y_{i+j+1}$ for $i\neq j$ and $i+j\leqslant m-2$. If $|G|=p^4$, it is $k(G)=0$. So, we can suppose $|G|\geqslant p^5$, whence $c(G)\in\{0, 1\}$. If $c(G)=1$, then we have $[Y_i, Y_j]=Y_{i+j+1}$ for all $i\neq j$ and (1.13) yields $k(G)=B_{m,p}(1)=0$. If $c(G)=0$, we get $k(G)=k(\bar{G})$ by using (1.5). Furthermore, $c(\bar{G})=1$ and $[\bar{Y}_i, \bar{Y}_j]=\bar{Y}_{i+j+1}$ whenever $i\neq j$, $i+j\leqslant m-2$, whence $k(\bar{G})=0$ and the theorem is proved. ∎

We note that the previous theorem determines the values of all the commutator subgroups of a p-group of maximal class satisfying $r(G)=f_0(|G|)$, since $[Y_i, Y_j]=[Y_i, Y_{i+1}]$ when $i=j$, $[Y_i, Y_j]=1$ for $i+j\geqslant m$ and the value of $[Y_i, Y_j]$ for $i+j=m-1$ is given in (1.9).

The following corollary completes Theorem (1.13), where the case $c(G)=0$ was lacking.

(2.2) COROLLARY. *Let G be a p-group of maximal class of order p^m such that $c(G)=0$. Then, $k(G)=B_{m,p}(c(G))$ if and only if $[Y_i, Y_j]=Y_{i+j+1}$ whenever $i\neq j$ and $i+j\leqslant m-2$.*

Proof. This result is a direct consequence of (2.1), since $c(G)=0$ implies $B_{m,p}(c(G))=0$. ∎

(2.3) THEOREM. *Let G be a p-group of maximal class of order $p^m=p^{2n+e}$ with $m\geqslant 5$ and suppose that $r(G)=f_0(|G|)$. Then, for any $\mathcal{M}$-chain of G, $G=H_0>H_1>\cdots>H_{m-4}$, we have that*

(1) *If* $i \in \{1, ..., n-2\}$ *and we set* $i = 2n'_i + e'_i$ *with* $e'_i \in \{0, 1\}$, *then*

$$k(H_i) = (n-i-1)\,F_{i-1}(p) - \alpha p^{i-1} + p^{e'_i}(eI_{2n'_i-2}(p) + J_{2n'_i-3}(p)) + e'_i n'_i,$$

where $\alpha = 0$ *or* 1 *according as* $c(G) \geqslant 1$ *or* $c(G) = 0$.

(2) *If* $i \in \{n-1, ..., m-4\}$ *and we set* $m - i = 2n_i + e_i$ *with* $e_i \in \{0, 1\}$, *then*

$$k(H_i) = e_i I_{2n_i-4}(p) + J_{2n_i-5}(p).$$

Proof. If $i \in \{n-1, ..., m-4\}$, then $Y_1(H_i) = Y_{i+1}(G) \leqslant Y_n(G)$. Hence, $Y_1(H_i)$ is abelian. Now, taking into account the remark before Theorem (1.6), we get that $k(H_i) = I_{m-i-5}(p) + J_{m-i-6}(p)$ or $J_{m-i-5}(p)$, according as $\exp|H_i| = m - i$ is odd or even. Hence, we deduce that $k(H_i) = e_i I_{2n_i-4}(p) + J_{2n_i-5}(p)$ in any case.

If $i \in \{1, ..., n-2\}$, we have already mentioned that $z_i = p^{i+1}$ and, consequently, (1.20) yields $k(H_i) = pk(H_{i-1}) - F_{i-1}(p) + n_{i-1} - \alpha_{i-1} - 1$. Hence, $k(H_1) = n - 2 - \alpha$ and

$$k(H_i) = pk(H_{i-1}) - F_{i-1}(p) + n - n'_{i-1} - e'_{i-1} + ee'_{i-1} - 1 \qquad (11)$$

for $i = 2, ..., n-2$, since $2n_{i-1} + e_{i-1} = m - (i-1) = 2n + e - 2n'_{i-1} - e'_{i-1} = 2(n - n'_{i-1} - e'_{i-1}(1-e)) + e(1 - 2e'_{i-1}) + e'_{i-1}$ and $e(1-2e'_{i-1}) + e'_{i-1} \in \{0, 1\}$ imply $n_{i-1} = n - n'_{i-1} - e'_{i-1}(1-e)$.

Now, we set $k(H_i) = (n-i-1)\,F_{i-1}(p) - \alpha p^{i-1} + W_i(p)$ for $i = 1, ..., n-2$. Using (11), we have

$$\begin{aligned}(n-i-1)\,&F_{i-1}(p) - \alpha p^{i-1} + W_i(p)\\ &= (n-i)\,pF_{i-2}(p) - \alpha p^{i-1} + pW_{i-1}(p) - F_{i-1}(p)\\ &\quad + n - n'_{i-1} - e'_{i-1} + ee'_{i-1} - 1.\end{aligned}$$

As $pF_{i-2}(p) = F_{i-1}(p) - 1$ and $i - 1 = 2n'_{i-1} + e'_{i-1}$, we deduce that

$$W_i(p) = pW_{i-1}(p) + n'_{i-1} + ee'_{i-1} \qquad \text{for} \quad i = 2, ..., n-2. \qquad (12)$$

We proceed to prove by induction on j that

$$W_{2j}(p) = eI_{2j-2}(p) + J_{2j-3}(p) \qquad \text{for} \quad j = 1, ..., [n/2] - 1. \qquad (13)$$

As $k(H_1) = n - 2 - \alpha$, $W_1(p) = 0$ and (12) yields $W_2(p) = e$. Hence, the assertion is true for $j = 1$. If it is true for j, we have

$$\begin{aligned}W_{2j+2}(p) &= pW_{2j+1}(p) + n'_{2j+1} + ee'_{2j+1}\\ &= p^2W_{2j}(p) + n'_{2j}\,p + ee'_{2j}\,p + n'_{2j+1} + ee'_{2j+1}\\ &= ep^2I_{2j-2}(p) + p^2J_{2j-3}(p) + jp + j + e\\ &= e(p^2I_{2j-2}(p) + 1) + p^2J_{2j-3}(p) + j(p+1)\\ &= eI_{2j}(p) + J_{2j-1}(p),\end{aligned}$$

and it is true for $j+1$. Now, (12) gives

$$W_{2j+1}(p)=p(eI_{2j-2}(p)+J_{2j-3}(p))+n'_{2j} \qquad \text{for} \quad 0\leqslant j\leqslant [(n-1)/2]-1. \tag{14}$$

We can unify (13) and (14) in the formula

$$\begin{aligned} W_i(p) &= p^{e'_i}(eI_{i-2-e'_i}+J_{i-3-e'_i}(p))+e'_i n'_{i-1} \\ &= p^{e'_i}(eI_{2n'_i-2}+J_{2n'_i-3}(p))+e'_i n'_i \qquad \text{for} \quad i=1, ..., n-2, \end{aligned}$$

where the last equality follows from the fact that $e'_i n'_{i-1}=e'_i(n'_i-1+e'_i)=e'_i n'_i-e'_i+e'^2_i=e'_i n'_i$, since $e'_i\in\{0, 1\}$ implies that $e'_i=e'^2_i$. Hence, we have proved that

$$k(H_i)=(n-i-1)\,F_{i-1}(p)-\alpha p^{i-1}+p^{e'_i}(eI_{2n'_i-2}(p)+J_{2n'_i-3}(p))+e'_i n'_i,$$

for $i=1, ..., n-2$. ∎

Remark. We note that the values of $k(H_i)$ for the case $c(G)=1$ could have been obtained directly just applying repeatedly Corollary (1.14), since $k(G)=0$ implies $k(G)=B_{m,p}(c(G))$. Nevertheless, the proof of (2.3) is not redundant, since we treat both cases $c(G)=0$ and $c(G)=1$ jointly (by means of the parameter α) and the case $c(G)=0$ cannot be deduced from (1.14).

(2.4) THEOREM. *Let G be a p-group of maximal class of order $p^m=p^{2n+e}$ and suppose that $r(G)=f_0(|G|)$. Then, we have that*

(1) *If $i\in\{1, ..., n-1\}$ and we set $i=2n'_i+e'_i$ with $e'_i\in\{0, 1\}$, then*

$$k(Y_i)=(n-i)\,F_{i-1}(p)-\alpha p^{i-1}+p^{e'_i}(eI_{2n'_i-2}(p)+J_{2n'_i-3}(p))+e'_i n'_i,$$

where $\alpha=0$ or 1 according as $c(G)\geqslant 1$ or $c(G)=0$.

(2) *If $i\in\{n, ..., m\}$ and we set $m-i=2n_i+e_i$ with $e_i\in\{0, 1\}$, then*

$$k(Y_i)=e_i I_{2n_i-2}(p)+J_{2n_i-3}(p).$$

Proof. If $i\in\{n, ..., m\}$, then $Y_i\leqslant Y_n$. Hence, Y_i is abelian and $r(Y_i)=|Y_i|=p^{m-i}=p^{2n_i+e_i}$. So, $p^{2n_i+e_i}=n_i(p^2-1)+p^{e_i}+k(Y_i)(p^2-1)(p-1)$, that is, $p^{e_i}(p^{2n_i}-1)=(p^2-1)(n_i+k(Y_i)(p-1))$. We deduce that $n_i+k(Y_i)(p-1)=p^{e_i}I_{2n_i-2}(p)=e_i(p-1)\,I_{2n_i-2}(p)+I_{2n_i-2}(p)$. Consequently, $k(Y_i)=e_i I_{2n_i-2}(p)+(I_{2n_i-2}(p)-n_i)/(p-1)=e_i I_{2n_i-2}(p)+J_{2n_i-3}(p)$.

If $i\in\{1, ..., n-1\}$, then $z_{i-1}=p^i+\alpha_{i-1}(p-1)$. Hence, (1.21) yields $k(Y_i)=pk(Y_{i-1})-F_{i-1}(p)+n_{i-1}-\alpha_{i-1}$. Now, arguing as in the proof of (2.3), we deduce the statement claimed. ∎

3. p-Groups of Maximal Class Satisfying $k(G)=1$

In this last section, we deal with the p-groups G of maximal class satisfying $r(G)=f_1(|G|)$ and we consider the same problems that were solved for $r(G)=f_0(|G|)$ in [14, 16] and in the preceding section. In fact, we find σ_G, the commutator subgroups, and the value of k for the subgroups of a $\mathscr{M}$-chain or the subgroups of the extended lower central series. We also give a result which is equivalent to determining the vector ∇_G. Before we begin to study these points, we want to make the following remark: whereas every p-group G verifying $r(G)=f_0(|G|)$ is a p-group of maximal class, that is not the case for the p-groups with $r(G)=f_1(|G|)$. As an example, the p-group $G=\langle a, b \mid a^{p^2}=1=b^{p^2}, a^b=a^{1+p}\rangle$ has order p^4 and $r(G)=p^3+p^2-p=f_1(p^4)$, but its nilpotency class is 2.

(3.1) Lemma. *Let G be a p-group of maximal class of order $p^m=p^{2n+e}$. Suppose that $r(G)=f_1(|G|)$ and $r(\bar{G})=f_0(|\bar{G}|)$, where $\bar{G}=G/Z(G)$. Then, $c(G)\geqslant 1$ and there exists $t\in\{1,...,n-2+e\}$ such that $z_t=p^{t+2}$ and $z_i=p^{i+1}$ for all $i\in\{1,...,n-1\}-\{t\}$. Moreover, t is determined by the property $[Y_t, Y_{m-t-2}]=1$.*

Proof. As G is a p-group of maximal class and $r(G)=f_1(|G|)$, we have $m\geqslant 5$. If $m\leqslant 6$, it follows from [16] that $|G|=p^5$, $\sigma_G=(p, p^3, p^2, p, 1)$, and $c(G)=3$. Furthermore, $n-2+e=n-1=1$ and $[Y_1, Y_2]=1$ show that the assertion of the theorem holds in this case.

Therefore, we can suppose $m\geqslant 7$. By using (1.15) we obtain $c(G)\leqslant 1$. Besides, $k(G)\neq k(\bar{G})$ and (1.5) imply $c(G)>0$. Consequently, $c(G)=1$. On the other hand, Lemma (1.2) gives

$$\sum_{i=1}^{n-2+e}\frac{z_i-\bar{z}_i}{p^{i+1}}=p-1. \tag{15}$$

It follows that there exist $t\in\{1,...,n-2+e\}$ such that $z_t>\bar{z}_t$. We note that $t<m-t-2$, since $t\leqslant n-2+e$. Taking into account (1.10), we deduce that $[Y_t, Y_{m-t-2}]=[Y_t, Y_{m-c(G)-t-1}]=1$. Hence, $z_t\geqslant |Y_{m-t-2}|=p^{t+2}$. As $k(\bar{G})=0$, we have $\bar{z}_i=p^{i+1}$ for $i=1,...,n-2+e$. So, $z_t\leqslant p\bar{z}_t=p^{t+2}$ and necessarily $z_t=p^{t+2}$. Now, (15) yields

$$\sum_{\substack{i=1\\ i\neq t}}^{n-2+e}\frac{z_i-\bar{z}_i}{p^{i+1}}=0,$$

whence $z_i=\bar{z}_i=p^{i+1}$ for $i=1,...,n-2+e$, $i\neq t$. For the theorem to be completely proved it only suffices to bear in mind that, when $e=0$, $c(G)=1$ implies $z_{n-1}=p^n$. ∎

(3.2) COROLLARY. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$. Suppose that $r(G) = f_1(|G|)$ and $r(\bar{G}) = f_0(|\bar{G}|)$. If $i \in \{1, ..., n-1\}$ and $x \in Y_i - Y_{i+1}$, then we have*

$$|C_{Y_i}(x)| = \begin{cases} p^{i+2}, & \textit{if} \quad i \neq t; \\ p^{t+3}, & \textit{if} \quad i = t. \end{cases}$$

Proof. If $i \neq t$ and $x \in Y_i - Y_{i+1}$, then $|C_{Y_i}(x)| \geqslant |\langle x \rangle Y_{m-i-1}| = p^{i+2}$, whence $|Cl_{Y_i}(x)| \leqslant p^{m-2i-2}$. So, $z_i = r_{Y_i}(s_i Y_{i+1}) \geqslant |s_i Y_{i+1}|/p^{m-2i-2} = p^{i+1}$. As (3.1) shows that $z_i = p^{i+1}$, it must be $|C_{Y_i}(x)| = p^{i+2}$ for all $x \in Y_i - Y_{i+1}$. The corresponding assertion for $i = t$ is proved in the same way just taking into account that $[Y_t, Y_{m-t-2}] = 1$. ▌

DEFINITION. Let G be a p-group of maximal class satisfying $r(G) = f_1(|G|)$. If $r(G/Y_w) = f_1(|G/Y_w|)$ and $r(G/Y_{w-1}) = f_0(|G/Y_{w-1}|)$, we say that Y_w is the *change residual* of G and represent it by G°.

We note that the previous definition makes sense, since every p-group of maximal class of order p^4 has $2p - 1 = f_0(p^4)$ conjugacy classes. It follows that, if $G^\circ = Y_w$, then $w \in \{5, ..., m\}$.

(3.3) THEOREM. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$. Suppose that $r(G) = f_1(|G|)$. Set $G^\circ = Y_w$ with $w = 2v + \varepsilon$ and $\varepsilon \in \{0, 1\}$. Then, we have*

(1) *$z_0 = p$ or $2p - 1$, according as $c(G) \geqslant 1$ or $c(G) = 0$.*

(2) *There exists $t \in \{1, ..., v - 2 + \varepsilon\}$ such that $z_t = p^{t+2}$ and $z_i = p^{i+1}$ for $i \in \{1, ..., n - 2 + e\} - \{t\}$. Furthermore, if $e = 0$, $z_{n-1} = p^{n-1}$ or p^n, according as $c(G) = 0$ or $c(G) \geqslant 1$.*

Proof. The possible values for z_0, as well as for z_{n-1} when $e = 0$, are well known. To prove the rest of the theorem we use induction on $|G^\circ|$. If $|G^\circ| = 1$, the assertion holds by virtue of (3.1). If $|G^\circ| > 1$, then $r(\bar{G}) = f_1(\bar{G})$ and $\bar{G}^\circ = \overline{G^\circ} = \bar{Y}_w = Y_w(\bar{G})$. So, the inductive hypothesis yields the existence of $t \in \{1, ..., v - 2 + \varepsilon\}$ such that $\bar{z}_t = p^{t+2}$ and $\bar{z}_i = p^{i+1}$ for $i \in \{1, ..., (n - 1 + e) - 2 + (1 - e)\} - \{t\} = \{1, ..., n - 2\} - \{t\}$. If $e = 1$, then $|\bar{G}| = p^{2n}$ and $c(\bar{G}) \geqslant 1$ imply $\bar{z}_{n-1} = p^n = p^{(n-1)+1}$. Thus, in any case, we have $\bar{z}_i = p^{i+1}$ for all $i \in \{1, ..., n - 2 + e\} - \{t\}$. Now, taking into account Corollary (1.4), which gives $z_j = \bar{z}_j$ for $j \in \{1, ..., n - 2 + e\}$, we obtain the result desired. ▌

(3.4) THEOREM. *Let G be a p-group of maximal class of order p^m with $r(G) = f_1(|G|)$. If $i, j \geqslant 1$ are such that $i < j$ and $i + j \leqslant m - 2$, then*

$$[Y_i, Y_j] = \begin{cases} Y_{i+j+1}, & \textit{if} \quad i \neq t; \\ Y_{i+j+1} \textit{ or } Y_{i+j+2}, & \textit{if} \quad i = t. \end{cases}$$

Proof. First, we note that $c(G/Z(G))\geqslant 1$ and $i+j\leqslant m-2$ imply $[Y_i, Y_j]\leqslant Y_{i+j+1}$. If $i=t$, $j\leqslant m-t-3$ and we suppose $[Y_t, Y_j]\leqslant Y_{t+j+3}$, in $\tilde{G}=G/Y_{t+j+3}$ we obtain $[\tilde{Y}_t, \tilde{Y}_j]=\tilde{1}$, whence $z_t\geqslant \tilde{z}_t\geqslant |\tilde{Y}_j|=p^{t+3}$, which is impossible. Hence, $[Y_t, Y_j]\in\{Y_{t+j+1}, Y_{t+j+2}\}$. If $i\neq t$, a similar reasoning yields $[Y_i, Y_j]=Y_{i+j+1}$, just taking into account that $z_i=p^{i+1}$ (note that $i<j$ and $i+j\leqslant m-2$ force $i\leqslant n-2+e$). ∎

DEFINITION. Let G be a p-group of maximal class of order p^m such that $r(G)=f_1(|G|)$ and $c(G)\geqslant 1$. If $i\in\{1, ..., m-1\}$, we say that $r_G(Y_i-Y_{i+1})$ takes its *normal value* when

$$r_G(Y_i-Y_{i+1})=\begin{cases} p^2-p, & \text{if } 1\leqslant i\leqslant n-1;\\ p-1, & \text{if } n\leqslant i\leqslant m-1.\end{cases}$$

We also define the tuple $\urcorner N(G)=(i_1, ..., i_s)$ formed by the indices i_j for which $r_G(Y_{i_j}-Y_{i_j+1})$ does not take its normal value and ordered so that $i_1<\cdots<i_s$.

(3.5) LEMMA. *Let G be a p-group of maximal class of order $p^m=p^{2n+e}$ with $r(G)=f_1(|G|)$ and $c(G)\geqslant 1$. If $\urcorner N(G)=(i_1, ..., i_s)$, then $s=2$ and $i_1=t$. Moreover, $r_G(Y_t-Y_{t+1})=p^3-p^2$ and $r_G(Y_{i_2}-Y_{i_2+1})=2p^2-3p+1$ or p^2-p, according as $i_2\leqslant n-1$ or $i_2\geqslant n$.*

Proof. Firstly, suppose that $i\leqslant t$. If $x\in Y_i-Y_{i+1}$ verifies $|C_G(x)|>|C_{Y_i}(x)|$, then there exists $g\in G-Y_i$ such that $g\in C_G(x)$. If $g\in G-Y_1$, then $x\in C_G(g)=\langle g\rangle Y_{m-1}$, impossible. Hence, $g\in Y_1-Y_i$. Let $j<i$ such that $g\in Y_j-Y_{j+1}$. Then, $\langle g\rangle\langle x\rangle Y_{m-j-1}\leqslant C_{Y_j}(g)$, whence $|C_{Y_j}(g)|\geqslant p^{j+3}$. But $j<t$ implies $|C_{Y_j}(y)|=p^{j+2}$ for all $y\in Y_j-Y_{j+1}$. Thus, we have arrived to a contradiction and it must be $|C_G(x)|=|C_{Y_i}(x)|$ for all $x\in Y_i-Y_{i+1}$ and for $i=1, ..., t$. That is,

$$|C_G(x)|=\begin{cases} p^{i+2}, & \text{if } i<t;\\ p^{t+3}, & \text{if } i=t.\end{cases}$$

It follows that $r_G(Y_i-Y_{i+1})=p^2-p$ for $i<t$ and that $r_G(Y_t-Y_{t+1})=p^3-p^2$. Consequently, $i_1=t$.

On the other hand, we have $\sum_{i=1}^{m-1} r_G(Y_i-Y_{i+1})-(\sum_{i=1}^{n-1}(p^2-p)+\sum_{i=n}^{m-1}(p-1))=f_1(|G|)-f_0(|G|)=p^3-p^2-p+1$. Hence, if we call δ_{i_j} the difference between $r_G(Y_{i_j}-Y_{i_j+1})$ and its normal value for $j=1, ..., s$, we obtain $\sum_{j=1}^{s}\delta_{i_j}=p^3-p^2-p+1$. As $r_G(Y_t-Y_{t+1})=p^3-p^2$, it is $\delta_{i_1}=p^3-2p^2+p$ and $\sum_{j=2}^{s}\delta_{i_j}=p^2-2p+1=(p-1)^2$. Consequently, $s\geqslant 2$. If we take $x\in Y_{i_2}-Y_{i_2+1}$, then $r_G(Y_{i_2}-Y_{i_2+1})=(p-1)\,r_G(xY_{i_2+1})=(p-1)(1+\lambda(p-1))$ for some $\lambda\geqslant 0$, since $r_G(xY_{i_2+1})\equiv 1 \pmod{p-1}$. We deduce that $\delta_{i_2}\geqslant(p-1)^2$. Therefore, $s=2$ and $\delta_{i_2}=(p-1)^2$, that is, $r_G(Y_{i_2}-Y_{i_2+1})=2p^2-3p+1$ or p^2-p, according as $i_2\leqslant n-1$ or $i_2\geqslant n$. ∎

(3.6) LEMMA. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$ with $r(G) = f_1(|G|)$ and $c(G) \geqslant 1$. Let $\tilde{G}$ be a quotient group of G of order $p^{2\tilde{n}+\tilde{e}}$ and such that $r(\tilde{G}) = f_1(|\tilde{G}|)$. If $\neg N(\tilde{G}) = (t, u)$ and $u \geqslant n$ or $u \leqslant \tilde{n} - 1$, then $\neg N(G) = (t, u)$.*

Proof. If $u \geqslant n$, then $u \geqslant \tilde{n}$ *and* $r_{\tilde{G}}(\tilde{Y}_u - \tilde{Y}_{u+1}) = p^2 - p$. Since $u \geqslant n$, it follows from the previous lemma that $r_G(Y_u - Y_{u+1}) = p^2 - p$ and $\neg N(G) = (t, u)$. The case when $u \leqslant \tilde{n} - 1$ is proved in a similar way. ∎

(3.7) LEMMA. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$ satisfying $r(G) = f_1(|G|)$ and $c(G) \geqslant 1$. Set $\bar{G} = G/Z(G)$ and suppose that $r(\bar{G}) = f_1(|\bar{G}|)$. If $\neg N(G) = (t, u)$ and $\neg N(\bar{G}) = (t, \bar{u})$, then one of the following conditions holds*:

(i) $u = \bar{u}$.

(ii) $\bar{u} = n - 1$, $u = m - t - 2$, *and* $e = 0$.

Proof. From (3.6) we obtain $u = \bar{u}$ in the cases $\bar{u} \geqslant n$ and $\bar{u} \leqslant (n - 1 + e) - 1$. Therefore, we can suppose that $n - 1 + e \leqslant \bar{u} \leqslant n - 1$. Then, necessarily $e = 0$ and $\bar{u} = n - 1$. If $i \in \{1, ..., m-3\}$, according to (3.4), we have $[Y_i, Y_{m-i-2}] = Y_{m-1}$ if $t \notin \{i, m-i-2\}$ and $i \neq m - i - 2$, that is, if $i \notin \{t, m-t-2, n-1\}$. In these cases, it follows from (1.10) that $r_G(Y_i - Y_{i+1}) = r_{\bar{G}}(\bar{Y}_i - \bar{Y}_{i+1})$. Hence, $r_G(Y_i - Y_{i+1}) = p^2 - p$ or $p - 1$ for $1 \leqslant i \leqslant n - 2$ and $i \neq t$, or $n \leqslant i \leqslant m - 1$ and $i \neq m - t - 2$, respectively. Consequently, $u \in \{n-1, m-t-2\}$, i.e., $u = n - 1 = \bar{u}$ or $u = m - t - 2$. ∎

Now, we can characterize the invariant u as follows

(3.8) THEOREM. *Let G be a p-group of maximal class of order p^m with $r(G) = f_1(|G|)$ and $c(G) \geqslant 1$. Set $\neg N(G) = (t, u)$. Then, u is the greatest integer smaller or equal than $m - t - 2$ satisfying $[Y_t, Y_u] = Y_{t+u+2}$.*

Proof. We argue by induction on $|G^\circ|$. If $|G^\circ| = 1$, (3.1) yields $[Y_t, Y_{m-t-2}] = 1$. Hence, $r_G(Y_{m-t-2} - Y_{m-t-1}) = p^2 - p$, $u = m - t - 2$ and the result is true. Suppose that $|G^\circ| > 1$. If $[Y_t, Y_{m-t-2}] = 1$, reasoning as above, the result trivially holds. If $[Y_t, Y_{m-t-2}] \neq 1$, then $r_G(Y_{m-t-2} - Y_{m-t-1}) = r_{\bar{G}}(\bar{Y}_{(m-1)-t-1} - \bar{Y}_{(m-1)-t}) = p - 1$ and $u \neq m - t - 2$. Now, (3.7) yields $u = \bar{u}$ and, from the inductive hypothesis, u is the greatest integer smaller or equal than $(m-1) - t - 2 = m - t - 3$ satisfying $[\bar{Y}_t, \bar{Y}_u] = \bar{Y}_{t+u+2}$. Taking into account (3.4), we deduce that $[Y_t, Y_u] = Y_{t+u+2}$ and $[Y_t, Y_i] = Y_{t+i+1}$ for $u + 1 \leqslant i \leqslant m - t - 3$. Since we have assumed $[Y_t, Y_{m-t-2}] \neq 1$, the assertion follows also in this case. ∎

In the next theorem, we are going to obtain an explicit formula for the number u. First, we need the following definition.

DEFINITION. Let G be a p-group of maximal class of order $p^m = p^{2n+e}$ with $r(G) = f_1(|G|)$. If $G^\circ = Y_w$ and t is the same as in (3.3), we define

$$T(\lambda) = 2^{\lambda-1}w - ((2^\lambda - 1)t + 2^\lambda) \qquad \text{for every} \quad \lambda \geqslant 1.$$

Clearly, the sequence $\{T(\lambda)\}_{\lambda \geqslant 1}$ satisfies the recursive relation

$$T(\lambda+1) = 2T(\lambda) - t.$$

From (3.3), part (2), we get $T(1) = w - t - 2 > t$, whence $T(\lambda) > t$ for all λ. Consequently, $T(\lambda+1) > T(\lambda)$ for all $\lambda \geqslant 1$. Thus, we can consider the number μ_0 defined as the smallest integer satisfying $T(\mu_0) \geqslant n$. With the above notation, we have:

(3.9) THEOREM. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$ with $r(G) = f_1(|G|)$ and $c(G) \geqslant 1$. Then, there exists $\mu \leqslant \mu_0$ such that $\neg N(G) = (t, T(\mu))$.*

Proof. We argue by induction on $|G^\circ|$. If $|G^\circ| = 1$, the result is true, since $u = m - t - 2 = w - t - 2 = T(1)$ and $\mu_0 = 1$. Suppose that $|G^\circ| > 1$. From the inductive hypothesis, we have $\bar{u} = T(\bar{\mu})$ with $\bar{\mu} \leqslant \bar{\mu}_0$. Taking into account (3.7), one of the following cases holds:

(i) $u = \bar{u}$.

(ii) $\bar{u} = n - 1$, $u = m - t - 2$, and $e = 0$.

In the first case, $u = T(\bar{\mu})$ and $\mu_0 = \bar{\mu}_0$ or $\bar{\mu}_0 + 1$. Consequently, $\mu = \bar{\mu}$ satisfies the conditions of the theorem. In the second case, bearing in mind that $\bar{\mu}_0$ is the smallest integer for which $T(\bar{\mu}_0) \leqslant n - 1$, we deduce that $\bar{\mu} = \bar{\mu}_0$. Thus, $n - 1 = \bar{u} = T(\bar{\mu}_0)$, whence $\mu_0 = \bar{\mu}_0 + 1$ and $u = m - t - 2 = 2n - t - 2 = 2(n-1) - t = 2T(\bar{\mu}_0) - t = T(\bar{\mu}_0 + 1)$. So, setting $\mu = \mu_0$, the theorem holds also in this case. ∎

(3.10) THEOREM. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$ with $r(G) = f_1(|G|)$ and $c(G) \geqslant 1$. Then $G - Y_1$ is made up of $p^2 - p$ conjugacy classes of order p^{m-2} and, if $\neg N(G) = (t, u)$, we have that:*

(1) *If $i \in \{1, ..., n-1\} - \{t, u\}$, $Y_i - Y_{i+1}$ is made up of $p^2 - p$ classes of order p^{m-i-2}.*

(2) *If $i \in \{n, ..., m-1\} - \{u\}$, $Y_i - Y_{i+1}$ is formed by $p - 1$ classes of order p^{m-i-1}.*

(3) *$Y_t - Y_{t+1}$ is formed by $p^3 - p^2$ classes of order p^{m-t-3}.*

(4) *If $u \leqslant n - 1$, $Y_u - Y_{u+1}$ is made up of $(p-1)^2$ classes of order p^{m-u-2} and $p^2 - p$ classes of order p^{m-u-3}. If $u \geqslant n$, then $Y_u - Y_{u+1}$ is formed by $p^2 - p$ classes of order p^{m-u-2}.*

Proof. The result for $G - Y_1$ is well known (cf. [1]). In order to prove the rest of the assertions of the theorem, we note that for every $x \in Y_i - Y_{i+1}$:

(1) If $i \in \{1, ..., n-1\} - \{t, u\}$, we have $r_G(Y_i - Y_{i+1}) = p^2 - p$. Now, $|C_G(x)| \geqslant |\langle x \rangle Y_{m-i-1}| = p^{i+2}$, i.e., $|Cl_G(x)| \leqslant p^{m-i-2}$, implies $|Cl_G(x)| = p^{m-i-2}$.

(2) If $i \in \{n, ..., m-1\}$, then $r_G(Y_i - Y_{i+1}) = p - 1$. In addition, $|C_G(x)| \geqslant |Y_{m-i-1}| = p^{i+1}$, i.e., $|Cl_G(x)| \leqslant p^{m-i-1}$, and necessarily $|Cl_G(x)| = p^{m-i-1}$.

(3) If $i = t$, we have $r_G(Y_t - Y_{t+1}) = p^3 - p^2$. Set $\tilde{G} = G/G^\circ$. Then, $|C_G(x)| \geqslant |C_{\tilde{G}}(\tilde{x})| \geqslant |\langle \tilde{x} \rangle \tilde{Y}_{w-t-2}| = p^{t+3}$, i.e., $|Cl_G(x)| \leqslant p^{m-t-3}$, whence $|Cl_G(x)| = p^{m-t-3}$.

(4) If $i = u = T(\mu)$ and $\tilde{G} = G/Y_{t+u+2}$, then $|C_G(x)| \geqslant |C_{\tilde{G}}(\tilde{x})| \geqslant |\tilde{Y}_t| = p^{u+2}$, since $[Y_t, Y_u] = Y_{t+u+2}$. That is, $|Cl_G(x)| \leqslant p^{m-u-2}$. We deduce that

$$|Cl_G(x)| = p^{m-u-2} \qquad \text{for all} \quad x \in Y_u - Y_{u+1}, \text{ if } u \geqslant n, \tag{16}$$

since, in that case, $r_G(Y_u - Y_{u+1}) = p^2 - p$.

On the other hand, arguing by induction on $|G^\circ|$ we get $|Cl_G(x)| \geqslant p^{m-u-3}$ for all $x \in Y_u - Y_{u+1}$. In fact, if $|G^\circ| = 1$, then $u = m - t - 2 \geqslant n$ and the result is true. Now, let $|G^\circ| > 1$. According to (16), we can suppose that $u \leqslant n - 1$. Since $m - t - 2 \geqslant n$, it follows from (3.7) that $u = \bar{u}$. If $u = n - 1$ and $e = 0$, then $\bar{u} = n - 1$ and $|\bar{G}| = p^{2(n-1)+1}$, whence $|Cl_G(x)| \geqslant |Cl_{\bar{G}}(\bar{x})| = p^{(m-1)-\bar{u}-2} = p^{m-u-3}$ and the assertion holds. Otherwise, we have $u \leqslant n - 2 + e$, whence $m \geqslant 7 = 2k + 5$ and $c(G) = 1$. We also have that $m - u - 2 \notin \{t, u\}$ (if $m - u - 2 = u$, then $m = 2u + 2 \leqslant 2(n - 2 + e) + 2 = 2n + e - (2 - e)$, impossible, and if $m - u - 2 = t$, then $u = m - t - 2 \geqslant 2n + e - (n - 2 + e) - 2 = n$, also impossible). Taking into account (3.4), we have $[Y_u, Y_{m-u-2}] \neq 1$, that is, $[Y_u, Y_{m-u-c(G)-1}] \neq 1$. Now, (1.10) part (1) yields $|Cl_G(x)| = p\,|Cl_{\bar{G}}(\bar{x})| \geqslant pp^{(m-1)-\bar{u}-3} = p^{m-u-3}$, as we wanted to see.

To finish the proof of the theorem, consider the case $u \leqslant n - 1$ and let λ_j be the number of conjugacy classes in $Y_u - Y_{u+1}$ of order $p^{m-u-2-j}$ $(j = 0, 1)$. Then, as $r_G(Y_u - Y_{u+1}) = 2p^2 - 3p + 1$, we have

$$\lambda_0 + \lambda_1 = 2p^2 - 3p + 1,$$

$$p^{m-u-2}\lambda_0 + p^{m-u-3}\lambda_1 = |Y_u - Y_{u+1}| = p^{m-u} - p^{m-u-1}.$$

Consequently, $\lambda_0 = (p-1)^2$ and $\lambda_1 = p^2 - p$, as we wanted to prove. ∎

(3.11) THEOREM. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$ such that $r(G) = f_1(|G|)$ and $c(G)=0$. Then $G - (Y_1 \cup C_G(Y_{m-2}))$ is made up of $(p-1)^2$ conjugacy classes of order p^{m-2} and $C_G(Y_{m-2}) - Y_2$ is formed by p^2-p classes of order p^{m-3}. Furthermore, if $\urcorner N(\bar{G}) = (t, u)$, we have that:*

(1) *If $i \in \{1, ..., n-2\} - \{t, u\}$, $Y_i - Y_{i+1}$ is made up of p^2-p classes of order p^{m-i-2}.*

(2) *If $i \in \{n-1, ..., m-1\} - \{u\}$, $Y_i - Y_{i+1}$ is formed by $p-1$ classes of order p^{m-i-1}.*

(3) *$Y_t - Y_{t+1}$ is formed by p^3-p^2 classes of order p^{m-t-3}.*

(4) *If $u \leqslant n-2$, $Y_u - Y_{u+1}$ is made up of $(p-1)^2$ classes of order p^{m-u-2} and p^2-p classes of order p^{m-u-3}. If $u \geqslant n-1$, then $Y_u - Y_{u+1}$ is formed by p^2-p classes of order p^{m-u-2}.*

Proof. The results for $G-(Y_1 \cup C_G(Y_{m-2}))$ and $C_G(Y_{m-2}) - Y_2$ can be found in [1] and [16], respectively. For the rest of the theorem, it suffices to combine the last theorem with (1.11), just taking into account that $c(\bar{G}) \geqslant 1$, and that $c(G)=0$ implies $|G| = p^{2n}$, whence $|\bar{G}| = p^{2(n-1)+1}$. ▌

The last two theorems can be used to obtain explicitly the vector ∇_G of a general p-group of maximal class satisfying $r(G) = f_1(|G|)$. Nevertheless, we will not give it in this paper, since the different possibilities for u make the expression of this result rather cumbersome. Anyway, with the information given in (3.10) and (3.11), one can derive ∇_G for any particular p-group of maximal class that he is handling, supposed $r(G) = f_1(|G|)$.

In the following theorem we precise the information given in (3.4) on the commutator subgroups of a p-group of maximal class with $k(G) = 1$.

(3.12) THEOREM. *Let G be a p-group of maximal class of order p^m with $r(G) = f_1(|G|)$. Then, $[Y_t, Y_j] = Y_{t+j+1}$ for every $j > t$ such that $t+j \leqslant m-2$ and $j \neq T(\lambda)$ for all $\lambda \geqslant 1$.*

Proof. We use induction on $|G^\circ|$. Take j verifying $t+j \leqslant m-2$ and $j \neq T(\lambda)$ for every $\lambda \geqslant 1$. If $|G^\circ| = 1$, it follows that $j \leqslant m-t-3$, since $j \neq T(1) = m-t-2$. Now, $r(\bar{G}) = f_0(|\bar{G}|)$ and (2.1) imply $[Y_t, Y_j] = Y_{t+j+1}$. If $|G^\circ| > 1$, the inductive hypothesis yields $[Y_t, Y_j] = Y_{t+j+1}$ for $j \leqslant m-t-3$. If $j = m-t-2$ and $[Y_t, Y_j] = Y_{t+j+2} = 1$, we have $r_G(Y_j - Y_{j+1}) = p^2-p$, since $j \geqslant n$. Consequently, using (3.9), there exists μ such that $j = T(\mu)$, a contradiction. ▌

To finish this section, in the following two theorems we determine the sequences $\{k(H_i)\}$ and $\{k(Y_i)\}$.

(3.13) THEOREM. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$ with $m \geq 5$ and suppose that $r(G) = f_1(|G|)$. Then, for any $\mathcal{M}$-chain of G, $G = H_0 > H_1 > \cdots > H_{m-4}$, we have that*

(1) *If $i \in \{1, ..., n-2\}$ and we write $i = 2n'_i + e'_i$ with $e'_i \in \{0, 1\}$, then*

$$k(H_i) = (n-i-1)\, F_{i-1}(p) + \beta_i p^i - \alpha p^{i-1} + p^{e'_i}(eI_{2n'_i-2}(p) + J_{2n'_i-3}(p)) + e'_i n'_i,$$

where $\alpha = 0$ or 1 according as $c(G) \geq 1$ or $c(G) = 0$ and $\beta_i = 0$ or 1 according as $i \geq t$ or $i < t$.

(2) *If $i \in \{n-1, ..., m-4\}$ and we set $m - i = 2n_i + e_i$ with $e_i \in \{0, 1\}$, then*

$$k(H_i) = e_i I_{2n_i-4}(p) + J_{2n_i-5}(p).$$

Proof. To prove this theorem we argue as in (2.3), just taking into account that $z_t = p^{t+2}$ and $z_i = p^{i+1}$ for $i \in \{1, ..., n-2\} - \{t\}$ yield $k(H_t) = pk(H_{t-1}) - p^t - F_{t-1}(p) + n_{t-1} - \alpha_{t-1} - 1$ and $k(H_i) = pk(H_{i-1}) - F_{i-1}(p) + n_{i-1} - \alpha_{i-1} - 1$ for $i \in \{1, ..., n-2\} - \{t\}$. ∎

(3.14) THEOREM. *Let G be a p-group of maximal class of order $p^m = p^{2n+e}$ and suppose that $r(G) = f_1(|G|)$. Then, we have that*

(1) *If $i \in \{1, ..., n-1\}$ and we write $i = 2n'_i + e'_i$ with $e'_i \in \{0, 1\}$, then*

$$k(Y_i) = (n-i)\, F_{i-1}(p) + \gamma_i p^i - \alpha p^{i-1} + p^{e'_i}(eI_{2n'_i-2}(p) + J_{2n'_i-3}(p)) + e'_i n'_i,$$

where $\alpha = 0$ or 1 according as $c(G) \geq 1$ or $c(G) = 0$ and $\gamma_i = 0$ or 1 according as $i > t$ or $i \leq t$.

(2) *If $i \in \{n, ..., m\}$ and we set $m - i = 2n_i + e_i$ with $e_i \in \{0, 1\}$, then*

$$k(Y_i) = e_i I_{2n_i-2}(p) + J_{2n_i-3}(p).$$

Proof. It suffices to reproduce the same reasoning of (2.4), with similar considerations as those made in the last theorem. ∎

REFERENCES

1. N. BLACKBURN, On a special class of p-groups, *Acta Math.* **100** (1958), 45–92.
2. W. BURNSIDE, "Theory of Groups of Finite Order," 2nd ed., Dover, New York, 1955.
3. P. X. GALLAGHER, The number of conjugacy classes in a finite group, *Math. Z.* **118** (1970), 175–179.
4. P. HALL, unpublished.

5. B. HUPPERT, "Endliche Gruppen I," Springer-Verlag, Berlin/Heidelberg/New York, 1967.
6. L. G. KOVÁCS AND C. R. LEEDHAM-GREEN, Some normally monomial p-groups of maximal class and large derived length, *Quart. J. Math. Oxford Ser. (2)* **37** (1986), 49–54.
7. M. LAZARD, Sur les groupes nilpotents et les anneaux de Lie, *Ann. Sci. École Norm. Sup. (3)* **71** (1954), 101–190.
8. C. R. LEEDHAM-GREEN AND S. MCKAY, On p-groups of maximal class, I, *Quart. J. Math. Oxford Ser. (2)* **27** (1976), 297–311.
9. C. R. LEEDHAM-GREEN AND S. MCKAY, On p-groups of maximal class, II, *Quart. J. Math. Oxford Ser. (2)* **29** (1978), 175–186.
10. C. R. LEEDHAM-GREEN AND S. MCKAY, On p-groups of maximal class, III, *Quart. J. Math. Oxford Ser. (2)* **29** (1978), 281–299.
11. R. J. MIECH, Metabelian p-groups of maximal class, *Trans. Amer. Math. Soc.* **152** (1970), 331–373.
12. R. J. MIECH, Some p-groups of maximal class, *Trans. Amer. Math. Soc.* **189** (1974), 1-47.
13. B. A. PANFEROV, Nilpotent groups with lower central factors of minimal ranks, *Algebra i Logika* **19** (1980), 701–706.
14. J. POLAND, Two problems on finite groups with k conjugate classes, *J. Austral. Math. Soc.* **8** (1968), 49–55.
15. R. SHEPHERD, Ph.D. Thesis, University of Chicago, 1971.
16. A. VERA-LÓPEZ AND B. LARREA, On p-groups of maximal class, *J. Algebra* **137** (1991), 77–116.